solution for L1 regularized optimization

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0.1 One dimensional example

Problem Statement:

$$\arg\min_x (x - a)^2 + \lambda |x|, \lambda > 0 \quad (1)$$

Solution:

0.1.1 Mirror symmetry property with respect to $a$

Suppose $\hat{x}$ is solution to $\arg\min_x (x - a)^2 - \lambda x$, then $-\hat{x}$ must be solution $\arg\min_x (x - (-a))^2 - \lambda x$. Because of this property, we can only focus on the case where $a \geq 0$.

0.1.2 Solution when $a \geq 0$

The function can be splitted into two parts:

$$f(x) = \begin{cases} (x - a)^2 - \lambda x = (x - (a + \lambda/2))^2 + a^2 - (a + \lambda/2)^2 & x \leq 0 \\ (x - a)^2 + \lambda x = (x - (a - \lambda/2))^2 + a^2 - (a - \lambda/2)^2 & x \geq 0 \end{cases}$$

When $x \leq 0$, the minimum value is $a^2$ when $\hat{x}$ is 0.
When $x \geq 0$, the solution depends on $a - \lambda/2$.

$$\hat{x} = \begin{cases} a - \lambda/2 & a - \lambda/2 \geq 0 \\ 0 & a - \lambda/2 < 0 \end{cases}$$

The minimum value for $x \geq 0$ is always less or equal that the minimum value for $x < 0$. When the minimum value for both condition is the same, they both give $\hat{x}$ to be 0. In summary, the solution is:

$$\hat{x} = (a - \lambda/2)_+, a \geq 0 \quad (2)$$

0.1.3 General solution without constraint on $a$

Based on the mirror symmetric property, when $a$ is less than 0. We can turn to solve the following problem based on last sub section and take a negative on the solution.

$$\arg\min_x (x - (-a))^2 + \lambda |x|$$

The solution to original problem would be

$$\hat{x} = -1 \times (a - \lambda/2)_+, a < 0 \quad (3)$$

Mering solution 2 and 3, we can get the final solution:

$$\hat{x} = \text{sign}(a) \times (|a| - \lambda/2)_+ \quad (4)$$